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Yukawa model on a lattice: Two-body states

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Abstract. We present some Quantum Field Theory (QFT) results concerning the Yukawa model, solved non-perturbatively with the help of lattice techniques. In particular we focus on the possibility of generating a two-nucleon bound state, as compared to the non-relativistic limit of the same model. Preliminary results show the appearance of zero modes of the Dirac operator. They limit the numerical solution of the model to values of the coupling constant which are too small to allow binding of the two-nucleon system.

PACS. 13.75.Cs Nucleon-nucleon interactions – 11.10.-z Field theory

1 Introduction

The nucleon-nucleon (NN) interaction is one of the most widely studied problems in theoretical physics. From meson exchange models [1,2] to effective chiral Lagrangians [3], a huge effort has been devoted to developing suitable NN potentials that could reproduce the nuclear binding energies and scattering properties. Using Green's function Monte Carlo methods one can nowadays compute the nuclear spectrum up to ~ 12 nucleons [2].

Most potential models are inspired by an underlying QFT, from which only a very particular kind of diagrams is taken into account when solving the dynamical equations: in practice the solution is currently possible only in the ladder approximation.

For the Wick-Cutkosky (WC) model, all crossed-ladder graphs were summed up in [4], and the resulting binding energies are much bigger than the ones obtained within the ladder approximation. This strong disagreement is one of the most important motivations for the present work. As the WC model is not consistent as a field theory [5], we will study the simplest renormalizable QFT involving fermions, where one species of fermions interacts with a scalar meson via a Yukawa coupling.

The interest of this approach is manifold. On the one hand, it allows a comparison with the results of the ladder approximation in various relativistic and non-relativistic (NR) equations. On the other hand, and including other couplings, it could provide a relativistic description of nuclear ground states in terms of the traditional degrees of freedom —mesons and nucleons— with no other restriction than those arising from the assumed structureless character of the constituents.

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2 The model

We consider a system of two identical fermions (ψ) interacting through the exchange of a scalar meson (ϕ) described by the Lagrangian density,

$$\mathcal{L} = \overline{\psi} D \psi + \mathcal{L}_{\mathrm{KG}}(\phi) + g_0 \overline{\psi} \phi \psi \,, \qquad (1)$$

where $D = \gamma_\mu \partial_\mu - M_0$ is the Dirac operator with a bare fermion mass M_0 and $\mathcal{L}_{\mathrm{KG}}$ is the Klein-Gordon Lagrangian for the scalar field. This model —with an additional $\lambda\phi^4$ term— has been widely studied in the framework of the Higgs phenomenology [6]. In the NR limit (1), it gives rise to the potential

$$V(r) = -\frac{g_0^2}{4\pi} \frac{e^{-m_s r}}{r} \,, \qquad (2)$$

where m_s is the meson mass. The NR model depends on a unique parameter, $G = \frac{g_0^2}{4\pi}\frac{M}{m_s}$, and the first bound state appears for $G \approx 1.68$. The existence of this unique scaling parameter holds in the Schrödinger equation but it is no longer true in a relativistic or a QFT description.

In order to study the bound states, one needs to take into account contributions to all orders in the coupling. A perturbative approach is therefore not suitable. Instead, a genuinely non-perturbative tool will be used: the lattice field theory. The Yukawa model is solved on a Euclidean space-time lattice, where vacuum expectation values are computed in the Feynman path integral approach. For the dressed nucleon propagator one has, for instance,

$$G^{\alpha\beta}(x,y) = \langle 0|\psi^\alpha(x)\overline{\psi}^\beta(y)|0\rangle = \frac{1}{Z}\int [\mathrm{d}\overline{\psi}][\mathrm{d}\psi][\mathrm{d}\phi]\psi^\alpha(x)\overline{\psi}^\beta(y)e^{-S_E(\overline{\psi},\psi,\phi)}, \quad (3)$$

where the Euclidean action acts as a probability distribution, allowing for a Monte Carlo integration.

We have chosen the following discretization of scalar fields:

$$S_{\mathrm{KG}} = \frac{1}{2}\sum_x \left[\left(8 + a^2 m_s^2\right)\phi_x^2 - 2\sum_\mu \phi_{x+\mu}\phi_x\right] \quad (4)$$

and for fermion ones:

$$S = \sum_{xy}\overline{\psi}_x D_{xy}\psi_y\,,$$

where D_{xy} is the Wilson-Dirac operator

$$D_{xy} = (1 + g_L\phi_x)\,\delta_{x,y} - \kappa\sum_\mu[(1-\gamma_\mu)\delta_{x+\hat{\mu},y} + (1+\gamma_\mu)\delta_{x-\hat{\mu},y}]\,, \quad (5)$$

in which the hopping parameter, $\kappa = 1/(8 + 2aM_0)$, and $g_L = 2\kappa g_0$ have been introduced.

Fermion fields, being Grassmann variables, have to be integrated out in an algebraic way, resulting in

$$G^{\alpha\beta}(x,y) = \frac{1}{Z}\int[\mathrm{d}\phi]D^{-1}{}^{\alpha\beta}_{xy}\det(D)e^{-S_{\mathrm{KG}}}\,. \quad (6)$$

This calculation is rather demanding in computing time due to the determinant. The task is considerably simplified in the "quenched" approximation, which consists in neglecting all virtual nucleon-antinucleon pairs originating from the meson field $\phi \to \bar{\psi}\psi$. Thanks to the heaviness of the nucleon, this appears to be a good approximation for the problem at hand and has been adopted all along this work. Note that this is not *a priori* justified for QCD, where quarks are very light. Nevertheless, the quenched approximation gives there qualitatively good results. Mathematically it amounts to setting $\det(D) = 1$. The main numerical task in calculating (6) is the inversion of the Dirac operator D_{xy}.

In the quenched approximation, and in the absence of meson self-interaction terms, the meson field is free, and ϕ field configurations can be independently generated by a Gaussian probability distribution in momentum space.

3 Spectrum of the Dirac operator

The spectrum of the free Dirac operator (5) lies inside a circle centered on $\lambda_0 = (1,0)$ with radius 8κ. In QCD when the interaction is turned on, the eigenvalues are modified,

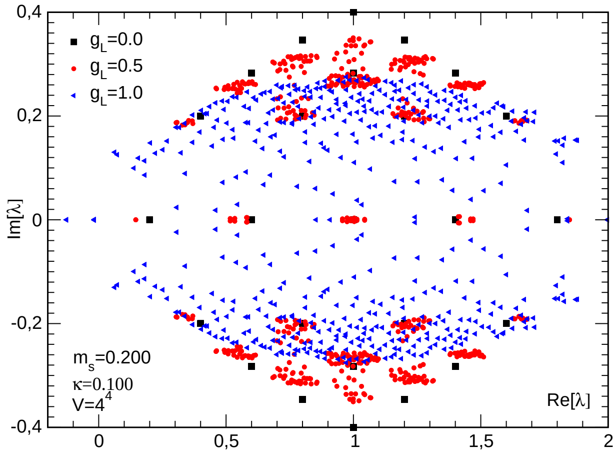

Fig. 1. Spectrum of the Wilson-Dirac operator for a small lattice and several values of the coupling.

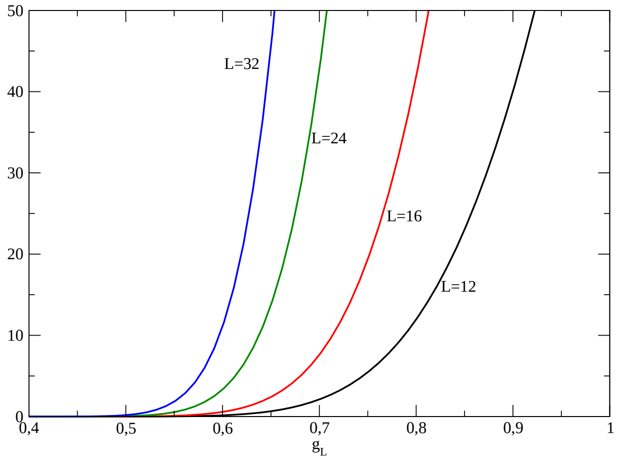

Fig. 2. Average number of negative eigenvalues for several lattices with $Lam_s = 5$ as a function of g_L.

but their real part is always positive. In the Yukawa model, on the contrary, the coupling term plays the role of a mass: as the coupling constant grows, the spectrum spreads out and some eigenvalues move to the $\mathrm{Re}(\lambda) < 0$ half-plane (fig. 1).

Eigenvalues with a negative real part spoil the convergence of most iterative algorithms, but this is not a fundamental problem. Nevertheless, for large values of the coupling and large lattices the probability of having very small eigenvalues grows dramatically. A simplified but significant picture can help to estimate the appearance of those small eigenvalues. The diagonal terms in (5) have the form $1 + g_L\phi_x$, which vanish for $\phi_x = -1/g_L$. According to (4), the values of ϕ_x are distributed around zero with a width:

$$\sigma^2 = \sum_{k\in V}\frac{1}{\hat{k}^2 + m_s^2}\,, \qquad \hat{k}^2 = 2\sum_\mu(1 - \cos(k_\mu)). \quad (7)$$

The average number of negative eigenvalues quickly increases with the coupling and the lattice size, as is shown in fig. 2. The non-diagonal terms in (5) sligthly modify this picture. In practice, for $L \sim 16$ the small eigenvalues

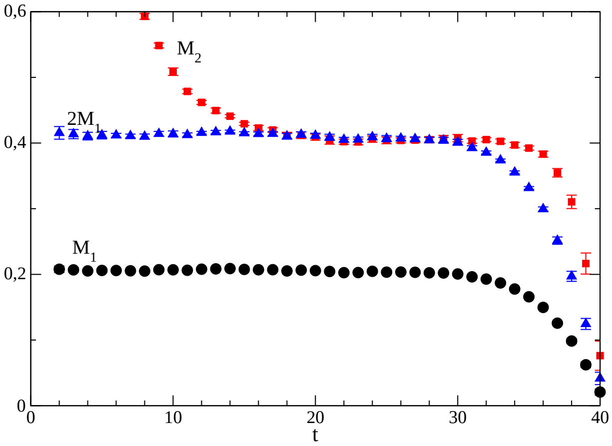

Fig. 3. One- (M_1) and two-body (M_2) effective masses *versus* the Euclidean time, for a $20^3 \times 80$ lattice with $g_0 \approx 1.6$ and $am_s = 0.200$. Two times fermion mass is plotted ($2M_1$) to compare with mass of two-fermion 0^+ state.

hinder the numerical solution of the linear system for $g_L \gtrsim 0.8$. This implies the existence of a maximum value of the coupling constant that can be used. The problem could perhaps be solved in the unquenched case, as the fermionic determinant would eliminate the configurations with very small $\det(D)$.

4 One- and two-body masses

The one-body mass is computed from the time-dependence of the Euclidean correlators:

$$C_1(t) = \sum_{\boldsymbol{x}} \mathrm{Tr}\left[G(\boldsymbol{x},t)\right] \sim e^{-M_1 t}\,, \qquad (8)$$

for large values of t, determining the fermion renormalized mass, M_1. Preliminary results on one-body masses for both scalar and pseudoscalar coupling were already presented in [7].

The two-body mass, M_2, is obtained in a similar way, from the time evolution of the $J(x) = \Gamma_{\alpha\beta}\psi_\alpha(x)\psi_\beta(x)$ operator, creating a nucleon pair

$$C_2(t) = \sum_{\boldsymbol{x}} \mathrm{Tr}\langle J(\boldsymbol{x},t)J^\dagger(\boldsymbol{0},0)\rangle \sim e^{-M_2 t}\,. \qquad (9)$$

For large values of t, it projects on the lowest energy state with the quantum numbers of $J(x)$. The matrix Γ determines the spin and parity of the state ($\Gamma = i\gamma_2\gamma_0\gamma_5$ for $J^\pi = 0^+$ —ground state— and $\Gamma = \gamma_2\gamma_0$ for $J^\pi = 0^-$). The exponential behavior is reached only at large values of t. It is useful to define an effective mass as

$$M_{\mathrm{eff}}(t) = \ln\left(\frac{C(t)}{C(t+1)}\right)$$

which, for large t, goes to the mass of the state and helps to find the adequate fitting window. Some results for one- and two-body masses can be found in fig. 3.

With this set of parameters, the two-fermion mass value is not distinguishable from twice the fermion mass. This is a common picture for the whole set of parameters tested and no signal of the existence of a bound state has been found below the critical value of the coupling constant[1].

5 Discussion

Renormalization effects have been analyzed for one-body masses, and the renormalization issues concerning the coupling constant were discussed in [7].

The existence of a maximum value of the coupling in a QFT treatment of the Yukawa model has been established. This critical value is smaller than the one needed to form a bound state in the NR limit, and no signal of such a bound state for lower couplings has been observed. The limitation on the coupling constant value is a consequence of the QFT approach. This is in contrast with the potential models where the coupling can usually take arbitrary large values.

The physical meaning of this result needs to be clarified. It may be related to the quenched approximation, or to the fact that we have neglected meson self-interactions. Note however that both approximations are performed in the non-relativistic treatment as well. For a given value of the lattice spacing, there are other ways to discretize the Yukawa model which do not have these zero modes. This has to be further studied and it is not clear that it would allow to reach larger renormalized coupling constants and in particular the bound-state regime. We are not yet in a position to decide whether the bound on the coupling constant we encounter is a lattice artefact or whether it really casts a doubt on the Yukawa theory itself.

It is known that the Yukawa theory is infrared free and, as such, encounters the "triviality problem" *i.e.* the fact that the ultraviolet cut-off cannot be driven to infinity without the theory becoming free. This means that it can only be considered as an effective theory with a physical ultraviolet cut-off. In addition, the problem encountered in inverting the Dirac operator appears also to put a restriction on the continuum limit. It is not clear whether both problems are related or just happen both to hinder the continuum limit.

Whether we manage or not to overcome the difficulty of reaching the domain where bound states appear, the connection between the QFT treatment and the Schrödinger approach can still be performed by calculating the low-energy scattering parameters thanks to the method proposed by Luscher [8] and recently considered in [9].

[1] There might exist bound states for very light mesons, near the Coulomb limit, but this regime is difficult to reach on the lattice due to the hierarchy of scales that appear.

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